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Classical relativistic spinning particle with anomalous magnetic moment: the precession of spin

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Abstract. The theory of classical relativistic spinning particles with c-number internal spinor variables, modelling accurately the Dirac electron, is generalized to particles with anomalous magnetic moments. The equations of motion are derived and the problem of spin precession is discussed and compared with other theories of spin.

1. Introduction

Although to this day the understanding of our first discovered and foremost elementary particle, the electron, remains far from being complete, it has in fact been advanced in recent years due to a consistent effort to unveil the content of a classical Lagrangian spinor theory for spinning particles reflecting rather accurately the properties of the Dirac electron. It is our purpose in the present work to generalize once more this theory and consider the interaction of a spinning electron, including its anomalous magnetic moment, with an external electromagnetic field.

The development of classical theories for spinning particles has a long history that goes back to Frenkel's [1] and Thomas's [2] first attempts to model the internal structure of the electron required to explain spectroscopic experiments. Similar equations were derived, although from different starting points, by Mattison [3], Bhabha [4] and Weyssenhoff and Raabe [5]. Around the same time that Thomas presented his theory, Dirac [6] also proposed his remarkable relativistic quantum mechanical equation for the electron, an equation that contained in it an accurate description of the spin. But as is well known spin is neither the result of relativity, nor of quantum mechanics. Therefore much work attempted to obtain spin precession equations as a classical limit of Dirac's equation. Work along these lines has been inconclusive as different classical equations are obtained according to the way the limits performed.

Later when laboratory experiments allowed us to measure the total magnetic moment of the electron, a classical relativistic equation of motion for spin precession was proposed for their interpretation by Bargmann *et al* (BMT) [7]. Generalizations of their equation to the case of inhomogeneous fields were described by Solomon [8], Nyborg [9], Barut [10] and Plathe [11] among others. But as shown by Bacry [12], the BMT equation is in fact equivalent to the Thomas's precession equation for the spinning electron.

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The phenomenological BMT equation was found by requiring a unique relativistic generalization of the rest frame precession equation for the spin together with additional constraints on the four-dimensional spin and velocity vector, and not starting from a Lagrangian as generally required in order to properly quantize the classical theory. Thus in general, theories obtained by conservation laws could not be consistently quantized, or when obtained from a Lagrangian, as Frenkel's theory, the problem of quantization has not been solved up to this day (see remarks by Ellis [13]). Moreover, the phenomenological constraints introduced by these theories may not represent the real electron. Another approach to this problem was reviewed by Mukunda *et al* [14]. In this line of development, group-theoretical methods were applied to the theory of extended objects including the spinning point particle. A drawback of this approach is that a Hamiltonian or mass operator was postulated in order to bypass Dirac's canonical formalism for constrained systems.

It has been argued that one should not be worried about quantization because the Dirac equation is an excellent description of the quantum electron, which is certainly true. But, as has been recently shown, the quantization of a classical spinning particle gives much more information than just Dirac's equation [15].

A quite different model for spinning particles has been studied by Barut and Zanghi [16]. In their model the magnitude of the spin tensor is not fixed nor is the magnitude of velocity as had been required in every previous theory of classical spinning particles. Here the spin vector is oscillatory even for a free particle and thus constant only on a temporal average basis. As in most previous theories, the orbital coordinate of the free particle describes a helical motion. But, in contrast to other theories, this result is not considered as a defect of the theory and as a matter of fact it is necessary for the particle to have an internal spin motion and also arises from Dirac's quantum equations [17]. The model has been properly quantized by canonical and Schrödinger quantization procedures [18], and by path integral methods [19]. It has been extended to curved spaces [20], to strings and membranes of any dimension [21]. Higher spin equations have been obtained in flat [15] and curved spaces [22], generalizing the usual Kemmer equation and providing a much more satisfactory derivation of these equations. Using this model for spin, the classical Lorentz–Dirac equation with radiation reaction has been generalized to include the radiative structure of the electron [23]. Consistent classical theories including radiation reaction have been obtained when in interaction with scalar, tensor and linearized gravity [24]. A Kaluza–Klein approach has given a compact description for the interaction of the electron with both, gravity and electromagnetism [25]. A complete symplectic geometric structure was found [16] and a modern differential geometric description in terms of complex fiber bundles exist [26]. Moreover, the internal geometry happens to be the same one that describes the quantum electron but already at a classical level. Finally, a classical formulation that parallels the quantum two body system was studied in detail [27]. As can be seen, one of the virtues of this model is its extensive generality, the others being its inner simplicity and completeness.

In this work we shall consider a classical charged point particle which interacts not only minimally with an electromagnetic field but also through a Pauli anomalous coupling to the electromagnetic field. The BMT equation has been previously obtained from a Lagrangian with this sort of term but the proportionality constant was adjusted to the total magnetic moment [10]. In the present case this constant will be just the anomalous magnetic moment as in Dirac's quantum equation and the spin will have a dynamical origin. The quantum version of this classical theory has already been constructed [28]. Here we shall obtain the equations of motion and therefore generalize previous precession equations for the spin of the electron with an anomalous magnetic moment.

2. The spinning particle and its minimal coupling

Let us start from the equations given by Barut and Zanghi [16] for a charged spinning particle. We write down the action principle in terms of an invariant worldline parameter τ to be associated later to the proper time of a well defined center of mass. The expression for the action reads

$$W_0=\int \mathrm{d}\tau\{-\mathrm{i}\lambda\bar{z}\dot{z}+p_\mu(\dot{x}^\mu-\bar{z}\gamma^\mu z)+e\bar{z}\gamma^\mu zA_\mu\}. \tag{1}$$

Here the variables z and $\bar{z}=z^+\gamma^0$ are four-component classical complex spinor variables which are functions of τ and represent the internal structure of the electron, i.e. the spin. The mass does not enter into this action but comes in as the value of the integral of motion $H=\bar{z}\gamma^\mu z\pi_\mu$, where $\pi_\mu=p_\mu-eA_\mu$ is the kinetic momentum.

A simplectic structure is given to the time development of the particle's motion by defining the Poisson bracket

$$\{f,g\}=\mathrm{i}\left\{\frac{\partial f}{\partial z}\frac{\partial g}{\partial \bar{z}}-\frac{\partial g}{\partial z}\frac{\partial f}{\partial \bar{z}}\right\}+g^{\mu\nu}\left\{\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial p^\nu}-\frac{\partial f}{\partial p^\mu}\frac{\partial g}{\partial x^\nu}\right\}. \tag{2}$$

Using this bracket it can easily be verified that z and $\mathrm{i}\bar{z}$ form a canonical set of conjugate variables. The pair of 'external' variables $\{x^\mu, p^\mu\}$, also form a set of conjugate variables which together with the four-component spinors form a complete set of dynamical variables for the electron.

The equations of motion that follow from this Lagrangian are ($\lambda=1$)

$$\begin{aligned} &\dot{\bar{z}}=-\mathrm{i}\bar{z}(p_\mu-eA_\mu)\gamma^\mu \qquad &&\dot{z}=\mathrm{i}(p_\mu-eA_\mu)\gamma^\mu z \\ &\dot{p}_\mu=eA_{\nu,\mu}\dot{x}^\nu \qquad &&\dot{x}_\mu=\bar{z}\gamma_\mu z. \end{aligned} \tag{3}$$

Now, by introducing the velocity $u_\mu\equiv\bar{z}\gamma_\mu z$, and the spin tensor $S_{\mu\nu}\equiv-\frac{i}{4}\bar{z}[\gamma_\mu,\gamma_\nu]z$, we get a different, but equivalent, complete set of dynamical equations

$$\begin{aligned} &\dot{x}_\mu=u_\mu \qquad &&\dot{u}_\mu=-4S^{\mu\rho}\pi_\rho \\ &\dot{S}_{\mu\nu}=u_\mu\pi_\nu-\pi_\mu u_\nu \qquad &&\dot{\pi}_\mu=eF_{\mu\rho}u^\rho \end{aligned} \tag{4}$$

where $F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}$ is the electromagnetic field.

While in previous theories for spinning particles the following conditions are usually phenomenologically assumed from the beginning,

$$u^\mu u_\mu=\text{constant} \qquad S_{\mu\nu}S^{\mu\nu}=\text{constant} \qquad u^\mu S_{\mu\nu}=0 \tag{5}$$

in this theory it can be verified that

$$\begin{aligned} &u^\mu u_\mu=(w_1^2+w_2^2) \\ &S_{\mu\nu}S^{\mu\nu}=(w_1^2-w_2^2)/2 \\ &u^\mu S_{\mu\nu}=+\mathrm{i}(w_2\eta_\nu)/2 \end{aligned} \tag{6}$$

where $\eta^\nu=-\mathrm{i}\bar{z}\gamma^5\gamma^\nu z$ and the quantities w_1 and w_2 are defined by $w_1=\bar{z}z$ and $w_2=\mathrm{i}\bar{z}\gamma^5 z$. Note that w_1 is a constant of the motion.

By direct differentiation of w_2 and after making use of the equations of motion one obtains the following second-order differential equation for w_2:

$$\ddot{w}_2 + 4\pi^2 w_2 = 2e\, \mathrm{i}\eta^\mu u^\nu F_{\nu\mu}. \tag{7}$$

As one can see the constraints (5) are only true in our case when $w_2 = 0$ which in general may hold only on the average.

A similar second-order equation can also be deduced for the spinor $\bar{z}$ (or z)

$$\ddot{\bar{z}} + \pi^2 \bar{z} = -\mathrm{i}e\bar{z}\gamma^\mu u^\nu F_{\mu\nu}. \tag{8}$$

The helical motion solution, also called Zitterbewegung, for a free particle then follows directly from this equation.

Let us look at the quantity

$$\Xi_{\alpha\beta\gamma} = \pi_\alpha S_{\beta\gamma} + \pi_\beta S_{\gamma\alpha} + \pi_\gamma S_{\alpha\beta} \tag{9}$$

considered previously by Barut and Zanghi [16]. In the presence of an external field we find that it obeys the equation of motion

$$\dot{\Xi}_{\alpha\beta\gamma} = eu^\eta (F_{\alpha\eta} S_{\beta\gamma} + F_{\beta\eta} S_{\gamma\alpha} + F_{\gamma\eta} S_{\gamma\alpha}). \tag{10}$$

Then, introducing $\Sigma_{\mu\nu} \equiv 4\pi_\rho \Xi^\rho_{\mu\nu}$ we obtain the equation of motion for the spin tensor $S_{\mu\nu}$:

$$\ddot{S}_{\mu\nu} + 4\pi^2 S_{\mu\nu} = +\Sigma_{\mu\nu} + eu^\alpha (F_{\nu\alpha} u_\mu - F_{\mu\alpha} u_\nu). \tag{11}$$

For a free particle, in particular, Ξ is a constant of the motion and the spin tensor has an oscillatory motion, just like its velocity or position.

We can now define a centre of mass by noting that only the total angular momentum $J_{\mu\nu}$ of the charge, defined by $J_{\mu\nu} = x_\nu \pi_\mu - x_\mu \pi_\nu + S_{\mu\nu}$, is conserved for a free particle. Writing $x_\mu = X_\mu + Q_\mu$, where X_μ is the centre of mass coordinate and Q_μ the internal relative coordinate position given by

$$Q_\mu = \frac{S_{\mu\alpha}\pi^\alpha}{\pi^2} \tag{12}$$

the total angular momentum $J_{\mu\nu}$ can also be cast into the form

$$J_{\mu\nu} = \bar{L}_{\mu\nu} + \bar{S}_{\mu\nu} \tag{13}$$

where $\bar{L}_{\mu\nu} = X_\nu \pi_\mu - X_\mu \pi_\nu$ can be interpreted as the orbital angular momentum of the centre of mass and $\bar{S}_{\mu\nu} = S_{\mu\nu} - Q_\mu \pi_\nu + Q_\nu \pi_\mu$ the spin of the centre of mass.

Since the first derivative of Q_μ is

$$\dot{Q}_\mu = u_\mu - \frac{m\pi_\mu}{\pi^2} + e\frac{S_{\mu\alpha}F^{\alpha\eta}u_\eta}{\pi^2} - 2e\frac{\pi F u}{\pi^3} S_{\mu\alpha}\pi^\alpha \tag{14}$$

where $\pi^2 \equiv \pi^\mu \pi_\mu$ and $\pi F u \equiv \pi_\alpha u_\beta F^{\alpha\beta}$, the relative position obeys a second-order equation which reduces to the equation of a harmonic oscillator when free of any external field.

3. Anomalous coupling and spin precession

We shall include into this theory the effect of the coupling between the anomalous magnetic moment of the particle and an external electromagnetic field by taking as a starting point the action

$$W = W_0 + \mu \int \mathrm{d}\tau\, \bar{z}[\gamma_\mu, \gamma_\nu] z F^{\mu\nu} \tag{15}$$

with μ a coefficient proportional to the anomalous magnetic moment and W_0 is as defined before. This constant is, for an electron, the result of the coupling to its self-field [29] but may be also an intrinsic property of the particle.

The equations of motion that follows from the action are ($\lambda = -1$)

$$\begin{aligned}
&\mathrm{i}\dot{\bar{z}} = -\bar{z}(p_\mu - eA_\mu)\gamma^\mu + \mu\bar{z}[\gamma_\mu, \gamma_\nu]F^{\mu\nu}\\
&-\mathrm{i}\dot{z} = -(p_\mu - eA_\mu)\gamma^\mu z + \mu[\gamma_\mu, \gamma_\nu]zF^{\mu\nu}\\
&\dot{p}_\mu = eA_{\nu,\mu}\dot{x}^\nu + \mu\bar{z}[\gamma_\alpha, \gamma_\beta]zF^{\alpha\beta}_{\ ,\mu}\\
&\dot{x}_\mu = \bar{z}\gamma_\mu z
\end{aligned} \tag{16}$$

or by using the variables $S_{\mu\nu}$, π_μ, u_μ and x_μ as before, we have

$$\begin{aligned}
&\dot{x}_\mu = u_\mu\\
&\dot{u}_\mu = +4S_{\mu\rho}\pi^\rho + 8\mathrm{i}\mu u^\alpha F_{\mu\alpha}\\
&\dot{S}_{\mu\nu} = -u_\mu\pi_\nu + \pi_\mu u_\nu - 8\mathrm{i}\mu\{S_{\nu\beta}F_\mu{}^\beta - S_{\mu\alpha}F_\nu{}^\alpha\}\\
&\dot{\pi}_\mu = eF_{\mu\rho}u^\rho + 4\mathrm{i}\mu S_{\alpha\beta}F^{\alpha\beta}_{\ ,\mu}\,.
\end{aligned} \tag{17}$$

From (4) it can be seen that $S_{\mu\nu}\pi^\nu \neq 0$ and from (6) that $S_{\mu\nu}u^\nu \neq 0$. Thus we cannot define the dual spin vector and its inverse either with π or with u alone. We therefore generalize the dual spin vector by using the linear combinations, $w_\mu = au_\mu + b\pi_\mu$, $w'_\mu = a'u_\mu + b'\pi_\mu$ where a, b, a', b' are constants. Then a definition for the spin vector would be

$$S_\mu = -\varepsilon_{\mu\alpha\beta\gamma}w^\alpha S^{\beta\gamma} \tag{18}$$

and its inverse

$$S_{\mu\nu} = \frac{1}{2ww'}\,\varepsilon_{\mu\nu\alpha\beta}w'^\alpha S^\beta \tag{19}$$

where $ww' = w^\mu w'_\mu$. As a result the equation of motion for the spin-four vector (choosing $a = 0$ for simplicity) reads for homogeneous fields

$$\dot{S}_\mu = \frac{e}{\pi w'}\,S_{[\mu}w'_{\alpha]}F^{\alpha\eta}u_\eta - 8\,\mathrm{i}\mu S^\beta F_{\beta\mu} + \frac{8\,\mathrm{i}\mu}{\pi w'}\,\{w'_{[\mu}S^{\beta]}F_{\beta\alpha}\pi^\alpha\} \tag{20}$$

where we indicate the antisymmetrization of indices by the square bracket.

We also give the result for the equation of motion for $\bar{S}_{\mu\nu}$, the spin thought to be attached to the centre of mass. For this quantity we obtain the equation

$$\begin{aligned}
\dot{\bar{S}}_{\mu\nu} = {}& eu^\alpha Q_{[\nu}F_{\mu]\alpha} + 4\mathrm{i}\mu Q_{[\nu}S_{\alpha\beta}F^{\alpha\beta}_{\ ,\mu]} - 8\mathrm{i}\mu S_{[\nu\alpha}F^{\ \alpha}_{\mu]} - 8\mathrm{i}\mu\pi^\alpha\pi_{[\mu}S_{[\alpha\beta}F_{\nu]]}{}^\beta\\
&+ \{eF^{\alpha\eta}u_\eta + 4\mathrm{i}\mu S_{\varepsilon\eta}F^{\varepsilon\eta,\alpha}\}\,\frac{\pi_{[\mu}S_{\nu]\alpha}}{\pi^2}\\
&- 2\{e\pi Fu + 4\mathrm{i}\mu\pi^\eta S_{\alpha\beta}F^{\alpha\beta}_{\ ,\eta}\}\,\frac{\pi^\alpha\pi_{[\mu}S_{\nu]\alpha}}{\pi^3}
\end{aligned} \tag{21}$$

with Q_μ as given in (12).

Equations (20) and (21) can be considered to be our main results as they constitute the analogue of the BMT equation

$$\dot{S}_{\mu\nu}=g\frac{e}{2m}F_{[\mu\beta}S^{\beta}{}_{\nu]}+(g-2)\frac{e}{2m}u_{[\mu}u_{\beta}F^{\alpha\beta}S_{\alpha\nu]}. \tag{22}$$

In terms of the dual vector defined by (if $u^2=1$)

$$S_\mu=\tfrac{1}{2}\varepsilon_{\mu\alpha\beta\gamma}u^\alpha S^{\beta\gamma}$$

the BMT equation becomes

$$\dot{S}_\mu=g\frac{e}{2m}F_{\mu\eta}S^\eta+(g-2)\frac{e}{2m}u_\mu u_\alpha F^{\beta\alpha}S_\beta. \tag{23}$$

Note that the third term in (21) proportional to μ has essentially the form of the BMT equation (22), however, in the latter, the coefficient is assumed to be the total magnetic moment $g=(2+a)$. The Dirac and Pauli couplings thus lead to different spin equations. The precession motion of the spin is, in fact, hidden in expression (20) due to the helical motion of the particle. Recall that for the quantum Dirac equation also the simple spin precession equation is obtained only after taking the matrix elements [30] and that the BMT equation does not hold for the exact operator form of the spin equations.

The limit of our equations (20) and (21) to the classical BMT equation (22) and (23) can be seen if we separate the rapidly oscillating zitterbewegung from the dynamical variables u_μ, π_μ and $S_{\mu\nu}$ by requiring

$$\overline{\pi_\mu S_\beta \pi_\alpha}=\bar{S}_\beta\bar{u}_\alpha\bar{u}_\mu \qquad \overline{S_\alpha\pi_\mu u_\eta}=\bar{S}_\alpha m\delta_{\mu\eta} \qquad \bar{\pi}^2=m^2 \tag{24}$$

and $\bar{u}^2=1$, $a'=0$, in (20). Here the bar means that we are taking an average over time. The result of this process is

$$\dot{\bar{S}}_\mu=\left(\frac{e}{m}+8\mathrm{i}\mu\right)F_{\mu\beta}\bar{S}^\beta+8\mathrm{i}\mu\{\bar{u}_\mu\bar{S}^\beta F_{\beta\alpha}\bar{u}^\alpha\}. \tag{25}$$

Thus after the identification $\{16\mathrm{i}\mu m\}/e=a$ where $a\equiv g-2$ is the anomalous magnetic moment, one obtains the BMT equation (22). It can be verified that equation of motion (21), in spite of its appearance, also has the form of a precession equation after performing the corresponding average. Finally, it should be noted that the dynamical variables appearing in the BMT equation correspond in our case to the temporal average of the velocity and the spin vector of the electron with zitterbewegung. The limiting procedure (24) will be discussed in detail elsewhere. Before any such averages, however, our exact spin precession equation (20) is

$$\dot{S}_\mu=\frac{e}{\pi w'}S_{[\mu}\pi_{\alpha]}F^{\alpha\eta}u_\eta+(g-2)\frac{e}{2\pi w'}[\pi_{[\mu}S^{\beta]}F_{\alpha\beta}\pi^\alpha+S^\beta F_{\mu\beta}].$$

4. Conclusions

To recapitulate, the coupling between the anomalous magnetic moment of a particle and external fields plays a prominent role in its time development. For instance, at short distances very strong interactions can arise due to the anomalous coupling to the electromagnetic field which for two- (or more) body quantum systems results in extremely close bound states corresponding to resonances as observed in nature [31]. It was therefore of interest to extend the single-event (from the point of view of the recently studied single-event theory of quantum mechanics [32] spinor theory for spinning particles with Dirac coupling to the more realistic situation of a spinning particle with anomalous magnetic moment interacting with an external field. We believe that the equations obtained here are the correct relativistic generalization for the motion of the spinning electron because we have started from a theory that does not contain arbitrary constraints. Moreover, as mentioned in the introduction, success of this model in describing a Dirac electron is impressive. It happens that the coupling between the internal and the external motions of the electron is just too subtle to be found by using only conservation laws without having started from a Lagrangian.

In a similar way that the non-relativistic limit of this theory gives the usual equation for the spin precession, a new quantum mechanical Pauli limit of Dirac's equation (which cannot be obtained from the square of the latter equation) follows from the application of these methods to the Dirac equation. This will be the subject of a future publication. We also plan to present elsewhere a consistent classical electrodynamics for particles with magnetic dipole moments, i.e. the derivation of radiation reaction terms for the equations of motion presented in this paper.

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References

[1] Frenkel W J 1926 *Z. Phys.* **37** 243
[2] Thomas L H 1927 *Phil. Mag.* **3** 1
[3] Mattison M 1937 *Acta Phys. Pol.* **6** 218
[4] Bhabha H 1940 *Proc. Ind. Acad. Sci.* **11** 246
[5] Weyssenhoff J and Raabe A 1947 *Acta Phys. Pol.* **9** 7
[6] Dirac P 1928 *Proc. R. Soc.* A **117** 610
[7] Bargmann V, Michel L and Telegdi V 1959 *Phys. Rev. Lett.* **2** 435
[8] Solomon A 1962 *Nuovo Cimento* **31** 1320
[9] Nyborg P 1964 *Nuovo Cimento* **31** 47
[10] Barut A O 1964 *Electrodynamics and the Classical Theory of Fields and Particles* (New York: Dover) p 74
[11] Plathe E 1966 *Nuovo Cimento* (Supp) **4** 246
[12] Bacry H 1962 *Nuovo Cimento* **26** 1164
[13] Ellis J R 1975 *Math. Proc. Camb. Phil. Soc.* **78** 145
[14] Mukunda N, Van Dam H and Biedenharn L C 1982 *Relativistic Models of Extended Hadrons Obeying a Mass-Spin Trajectory Constraint* Lecture Notes in Physics (New York: Springer)
[15] Barut A O 1990 *Phys. Lett.* **237B** 436
[16] Barut A O and Zanghi N 1984 *Phys. Rev. Lett.* **52** 2009
[17] Barut A O and Thacker J 1985 *Phys. Rev.* D **31** 1386
[18] Barut A O and Pavsic M 1987 *Class. Quantum Grav.* **2** L131
[19] Barut A O and Duru I 1989 *Phys. Rep.* **172** 1
[20] Barut A O and Pavsic M 1987 *Class. Quantum Grav.* **4** 141

[21] Barut A O and Pavsic M 1992 *Mod. Phys. Lett.* A **7** 1381
[22] Barut A O and Cruz M G 1992 to be published
[23] Barut A O and Unal N 1989 *Phys. Rev.* A **40** 5404
[24] Barut A O and Cruz M G 1993 *Phys. Lett.* **308B** 247
[25] Barut A O and Pavsic M 1988 *Class. Quantum Grav.* **5** 707
[26] Rawnsley J 1992 *Lett. Math. Phys.* **24** 331
[27] Barut A O, Onem C and Unal U 1990 *J. Phys. A: Math. Gen.* **23** 1113
[28] Barut A O and Alan A 1992 *Europhys. Lett.* **17** 119
[29] Barut A O and Dowling 1989 *J. Z. Naturforsch.* **44a** 1051
[30] Sokolov A A and Ternov I M 1986 *Radiation from Relativistic Electrons* (New York: American Institute of Physics Translation Series) p 170
[31] Barut A O and Kraus J 1976 *J. Math. Phys.* **17** 506
[32] Barut A O 1990 *Found. Phys.* **20** 1233